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# UNIVERSITÀ DEGLI STUDI DI TORINO

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## LINEAR ALGEBRA AND TORIC DATA OF WEIGHTED PROJECTIVE SPACES

**Abstract.** This paper is devoted to give characterizations of suitable matrices associated with fans and polytopes defining a weighted projective space and switching rules between them.

### Introduction

The aim of the present paper is to give characterizations of fans and polytopes defining a weighted projective space (*wps* for short) and switching rules between them, from a linear algebraic point of view.

After recalling some notation and preliminaries on toric varieties, the starting point here is the usual definition of a (complex) *wps* as a geometric quotient (see Definition 2), directly checking its natural toric structure. The bridge with the classical presentation of toric varieties via fans is then given by the Cox Theorem [8] Thm. 2.1.

Section 2 is devoted to the characterization of a *wps*'s fan: up to permutations on generators it is possible to associate a  $n \times (n + 1)$  integer matrix with the fan of a  $n$ -dimensional *wps* (so called *fan matrix*) whose entries turn out to verify an amount of relations (see equivalent conditions in the Theorem 3, giving a linear algebraic characterization of a *wps*'s fan). Until here almost nothing is new, since the equivalence of conditions (1) and (2) in Theorem 3 can be recovered from [4] and [2], while condition (3) can be deduced from [7] Thm. 3.6. Anyway, we were not able to find in the literature the relations between the fan generators  $\mathbf{v}_j$ 's of a *wps*  $\mathbb{P}(Q)$ , the associated primitive vectors  $\mathbf{n}_j$ 's and the reduction  $Q'$  of the weight vector  $Q$ , as explained in Lemma 1, although probably well-known to the experts. Notice that Lemma 1 can be used as a key step to get a completely combinatoric proof of the well-known Reduction Theorem 2 (see [18] Thm. 1.26 and its proof). Probably the most original result in Section 2 is the Proposition 5 where it is shown that the fan of a given *wps*  $\mathbb{P}(Q)$  is *encoded* in the switching matrix giving the *Hermite normal form* (HNF for short) of the transposed weight vector  $Q^T$ . This section ends up with the Proposition 6 in which on the one hand (parts from (1) to (3)) we rewrite the Conrads's presentation of a *wps*'s fan matrix, but proved by directly starting from relations given in the Theorem 3, and on the other hand (part (4)) we present a *Q-canonical* form for the fan of  $\mathbb{P}(Q)$ , only depending on the weights order in  $Q$  (see Remark 4), which can be simply obtained by the HNF of *any* fan matrix of  $\mathbb{P}(Q)$ . In our opinion this Proposition describes a clean and easy method to get a fan of  $\mathbb{P}(Q)$  even by hand (see Example 1).

Section 3 is dedicated to characterize polytopes associated with a polarized *wps*.

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As far as we know, results of this section were not known before. Let  $O(1)$  be the *minimal* polarization given by a generator of the Picard group  $\text{Pic}(\mathbb{P}(Q))$ . Then we draw a *fan-polytope correspondence* between fans of  $\mathbb{P}(Q)$  and polytopes of  $(\mathbb{P}(Q), O(1))$ : this is given, up to suitable weightings, in the one direction by taking the transposed inverse (so called *transverse*) of a maximal submatrix of a fan matrix, in the other direction by an obvious completion of the transposed adjoint matrix of the polytope matrix (see Definitions 5 and 7 and Remark 10). In our opinion this correspondence combined with the previous Proposition 6 provides a clean and easy method to get a polytope of  $(\mathbb{P}(Q), O(m))$ , even by hand, up to the elementary computation of the inverse of a (possibly big) matrix (see Example 2). Main results of this section are given by the Theorem 4, which is a direct consequence of Lemma 1, and the Proposition 9. This section ends up with the Theorem 5 giving a linear algebraic characterization of a polarized wps's polytope. This result has to be thought of as the polytopal counterpart of Theorem 3.

## 1. Preliminaries and notation

### 1.1. Toric varieties

A *n-dimensional toric variety* is an algebraic normal variety  $X$  containing the *torus*  $T := (\mathbb{C}^*)^n$  as a Zariski open subset such that the natural multiplicative self-action of the torus can be extended to an action  $T \times X \rightarrow X$ .

Let us quickly recall the classical approach to toric varieties by means of *cones* and *fans*. For proofs and details the interested reader is referred to the extensive treatments [11], [15], [17] and the recent and quite comprehensive [10].

As usual  $M$  denotes the *group of characters*  $\chi : T \rightarrow \mathbb{C}^*$  of  $T$  and  $N$  the *group of 1-parameter subgroups*  $\lambda : \mathbb{C}^* \rightarrow T$ . It follows that  $M$  and  $N$  are  $n$ -dimensional dual lattices via the pairing

$$\begin{aligned} M \times N &\longrightarrow \text{Hom}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{C}^* \\ (\chi, \lambda) &\longmapsto \chi \circ \lambda \end{aligned}$$

which translates into the standard pairing  $\langle u, v \rangle = \sum u_i v_i$  under the identifications  $M \cong \mathbb{Z}^n \cong N$  obtained by setting  $\chi(\mathbf{t}) = \mathbf{t}^{\mathbf{u}} := \prod t_i^{u_i}$  and  $\lambda(t) = t^{\mathbf{v}} := (t^{v_1}, \dots, t^{v_n})$ .

### Cones and affine toric varieties

Define  $N_{\mathbb{R}} := N \otimes \mathbb{R}$  and  $M_{\mathbb{R}} := M \otimes \mathbb{R} \cong \text{Hom}(N, \mathbb{Z}) \otimes \mathbb{R} \cong \text{Hom}(N_{\mathbb{R}}, \mathbb{R})$ .

A *convex polyhedral cone* (or simply a *cone*)  $\sigma$  is the subset of  $N_{\mathbb{R}}$  defined by

$$\sigma = \langle \mathbf{v}_1, \dots, \mathbf{v}_s \rangle := \{r_1 \mathbf{v}_1 + \dots + r_s \mathbf{v}_s \in N_{\mathbb{R}} \mid r_i \in \mathbb{R}_{\geq 0}\}$$

The  $s$  vectors  $\mathbf{v}_1, \dots, \mathbf{v}_s \in N_{\mathbb{R}}$  are said to *generate*  $\sigma$ . A cone  $\sigma = \langle \mathbf{v}_1, \dots, \mathbf{v}_s \rangle$  is called *rational* if  $\mathbf{v}_1, \dots, \mathbf{v}_s \in N$ , *simplicial* if  $\mathbf{v}_1, \dots, \mathbf{v}_s$  are  $\mathbb{R}$ -linear independent and *non-singular* if  $\mathbf{v}_1, \dots, \mathbf{v}_s$  can be extended by  $n - s$  further elements of  $N$  to give a basis of the lattice  $N$ .

A cone  $\sigma$  is called *strictly convex* if it does not contain a linear subspace of positive dimension of  $N_{\mathbb{R}}$ .

The *dual cone*  $\sigma^{\vee}$  of  $\sigma$  is the subset of  $M_{\mathbb{R}}$  defined by

$$\sigma^{\vee} = \{\mathbf{u} \in M_{\mathbb{R}} \mid \forall \mathbf{v} \in \sigma \quad \langle \mathbf{u}, \mathbf{v} \rangle \geq 0\}$$

A *face*  $\tau$  of  $\sigma$  (denoted by  $\tau < \sigma$ ) is the subset defined by

$$\tau = \sigma \cap \mathbf{u}^{\perp} = \{\mathbf{v} \in \sigma \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$$

for some  $\mathbf{u} \in \sigma^{\vee}$ . Observe that also  $\tau$  is a cone.

Gordon's Lemma (see [15] §1.2, Proposition 1) ensures that the semigroup  $S_{\sigma} := \sigma^{\vee} \cap M$  is *finitely generated*. Then also the associated  $\mathbb{C}$ -algebra  $A_{\sigma} := \mathbb{C}[S_{\sigma}]$  is finitely generated. A choice of  $r$  generators gives a presentation of  $A_{\sigma}$

$$A_{\sigma} \cong \mathbb{C}[X_1, \dots, X_r] / I_{\sigma}$$

where  $I_{\sigma}$  is the ideal generated by the relations between generators. Then

$$U_{\sigma} := \mathcal{V}(I_{\sigma}) \subset \mathbb{C}^r$$

turns out to be an *affine toric variety*. In other terms an affine toric variety is given by  $U_{\sigma} := \text{Spec}(A_{\sigma})$ . Since a closed point  $x \in U_{\sigma}$  is an evaluation of elements in  $\mathbb{C}[S_{\sigma}]$  satisfying the relations generating  $I_{\sigma}$ , then it can be identified with a semigroup morphism  $x : S_{\sigma} \rightarrow \mathbb{C}$  assigned by thinking of  $\mathbb{C}$  as a multiplicative semigroup. In particular the *characteristic morphism*

$$(1) \quad \begin{array}{ccc} x_{\sigma} : \sigma^{\vee} \cap M & \longrightarrow & \mathbb{C} \\ \mathbf{u} & \longmapsto & \begin{cases} 1 & \text{if } \mathbf{u} \in \sigma^{\perp} \\ 0 & \text{otherwise} \end{cases} \end{array}$$

which is well defined since  $\sigma^{\perp} < \sigma^{\vee}$ , defines a *characteristic point*  $x_{\sigma} \in U_{\sigma}$  whose torus orbit  $O_{\sigma}$  turns out to be a  $(n - \dim(\sigma))$ -dimensional torus embedded in  $U_{\sigma}$  (see e.g. [15] §3).

### Fans and toric varieties

A *fan*  $\Sigma$  is a finite set of cones  $\sigma \subset N_{\mathbb{R}}$  such that

1. for any cone  $\sigma \in \Sigma$  and for any face  $\tau < \sigma$  then  $\tau \in \Sigma$ ,
2. for any  $\sigma, \tau \in \Sigma$  then  $\sigma \cap \tau < \sigma$  and  $\sigma \cap \tau < \tau$ .

For every  $i$  with  $0 \leq i \leq n$  denote by  $\Sigma(i) \subset \Sigma$  the subset of  $i$ -dimensional cones, called the  *$i$ -skeleton* of  $\Sigma$ . A fan  $\Sigma$  is called *simplicial* if every cone  $\sigma \in \Sigma$  is simplicial and *non-singular* if every such cone is non-singular. The *support* of a fan  $\Sigma$  is the subset  $|\Sigma| \subset N_{\mathbb{R}}$  obtained as the union of all of its cones i.e.

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}}.$$

If  $|\Sigma| = N_{\mathbb{R}}$  then  $\Sigma$  will be called *complete* or *compact*.

Since for any face  $\tau < \sigma$  the semigroup  $S_{\sigma}$  turns out to be a sub-semigroup of  $S_{\tau}$ , there is an induced immersion  $U_{\tau} \hookrightarrow U_{\sigma}$  between the associated affine toric varieties which embeds  $U_{\tau}$  as a principal open subset of  $U_{\sigma}$ . Given a fan  $\Sigma$  one can construct an associated toric variety  $X(\Sigma)$  by patching all the affine toric varieties  $\{U_{\sigma} \mid \sigma \in \Sigma\}$  along the principal open subsets associated with any common face. Moreover for every toric variety  $X$  there exists a fan  $\Sigma$  such that  $X \cong X(\Sigma)$  (see [17] Theorem 1.5). It turns out that ([17] Theorems 1.10 and 1.11; [15] §2):

- $X(\Sigma)$  is non-singular if and only if the fan  $\Sigma$  is non-singular;
- $X(\Sigma)$  is complete if and only if the fan  $\Sigma$  is complete.

In the following a 1-generated fan  $\Sigma$  is a fan generated by a set of  $n+1$  integral vectors i.e. a fan whose cones  $\sigma \subset N \otimes \mathbb{R}$  are generated by any proper subset of a given finite subset  $\{\mathbf{v}_0, \dots, \mathbf{v}_n\} \subset N$ : we will write

$$(2) \quad \Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) .$$

Given a 1-generated fan  $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$ , the matrix  $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$  will be called a *fan matrix* of  $\Sigma$ . Notice that  $\Sigma$  determines  $V$  up to a permutations of columns, meaning that  $\Sigma$  admits  $(n+1)!$  associated fan matrices.

If  $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$  is a fan matrix of  $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$  then we will denote the maximal square sub-matrices of  $V$  and the associated  $n$ -minors as follows

$$(3) \quad \forall 0 \leq j \leq n \quad V^j := (\mathbf{v}_0, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_n) , \quad V_j = \det(V^j) .$$

### Polytopes and projective toric varieties

A *polytope*  $\Delta \subset M_{\mathbb{R}}$  is the convex hull of a finite set of points. If this set is a subset of  $M$  then the polytope is called *integral*. Starting from an integral polytope one can construct a projective toric variety as follows. Here we will follow the approach of [1], which the interested reader is referred to for proofs and details (see also [9] §3.2.2).

For any  $k \in \mathbb{N}$  one can define the dilated polytope  $k\Delta := \{k\mathbf{u} \mid \mathbf{u} \in \Delta\}$ . It is then possible to define a graded  $\mathbb{C}$ -algebra  $S_{\Delta}$ , associated with the integral polytope  $\Delta$ , as follows. For any  $\mathbf{u} \in k\Delta \cap M$  consider the associated character  $\chi^{\mathbf{u}} : \mathbf{t} \mapsto \mathbf{t}^{\mathbf{u}}$ . Given  $t \in \mathbb{C}^*$  consider the *monomial*  $t^k \chi^{\mathbf{u}} : \mathbf{t} \mapsto t^k \mathbf{t}^{\mathbf{u}}$ . It well defines a *monomial product*  $t^{k_1} \chi^{\mathbf{u}_1} \cdot t^{k_2} \chi^{\mathbf{u}_2} := t^{k_1+k_2} \chi^{\mathbf{u}_1+\mathbf{u}_2}$  where  $\mathbf{u}_1 + \mathbf{u}_2 \in (k_1 + k_2)\Delta$ . Let  $S_{\Delta}$  be the  $\mathbb{C}$ -algebra generated by all monomials  $\{t^k \chi^{\mathbf{u}} \mid k \in \mathbb{N}, \mathbf{u} \in k\Delta\}$  which is a graded object by setting  $\deg(t^k \chi^{\mathbf{u}}) = k$ .

The *projective variety*  $\mathbb{P}_{\Delta} := \text{Proj}(S_{\Delta})$  turns out to be naturally a *toric variety* whose fan  $\Sigma_{\Delta}$  can be recovered as follows. For any nonempty face  $F \prec \Delta$  consider the cone

$$\check{\sigma}_F := \{r(\mathbf{u} - \mathbf{u}') \mid \mathbf{u} \in \Delta, \mathbf{u}' \in F, r \in \mathbb{R}_{\geq 0}\} \subset M_{\mathbb{R}}$$

and define  $\sigma_F := \check{\sigma}_F^{\vee} \subset N_{\mathbb{R}}$ . Then  $\Sigma_{\Delta} := \{\sigma_F \mid F \prec \Delta\}$  turns out to be a fan, called the *normal fan* of the polytope  $\Delta$ , such that there exists a very ample divisor  $H$  of  $X(\Sigma_{\Delta})$  for

which  $(X(\Sigma_\Delta), H) \cong (\mathbb{P}_\Delta, O(1))$ , where  $O(1)$  is the natural polarization of  $\mathbb{P}_\Delta = \text{Proj}(S_\Delta)$  (see [1] Proposition 1.1.2).

Viceversa a *projective toric variety* is the couple  $(X(\Sigma), H)$  of a toric variety  $X(\Sigma)$  and a polarization given by (the linear equivalence class of) a hyperplane section  $H$ . For any 1-cone  $\rho \in \Sigma(1)$ , consider the torus stable divisor  $D_\rho := \overline{O}_\rho$  defined as the closure of the torus orbit of the characteristic point  $x_\rho$ , defined in (1). Since those divisors generate the Chow group of Weil divisors  $A_{n-1}(X(\Sigma))$  (see [15] §3.4), there exist integer coefficients  $a_\rho \in \mathbb{Z}$  such that  $H = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ . It is then well defined the integral polytope

$$(4) \quad \Delta_H := \{\mathbf{u} \in M_\mathbb{R} \mid \forall \rho \in \Sigma(1) \langle \mathbf{u}, \mathbf{n}_\rho \rangle \geq -a_\rho\}$$

where  $\mathbf{n}_\rho$  is the unique generator of the semigroup  $\rho \cap N$ . Then

$$(\mathbb{P}_{\Delta_H}, O(1)) \cong (X(\Sigma), H).$$

## 1.2. Hermite normal form

It is well known that Hermite algorithm provides an effective way to determine a basis of a subgroup of  $\mathbb{Z}^n$ . We briefly recall the definition and the main properties. For details, see for example [6].

**DEFINITION 1.** *An  $m \times n$  matrix  $M = (m_{ij})$  with integral coefficients is in Hermite normal form (abbreviated HNF) if there exists  $r \leq m$  and a strictly increasing map  $f : \{1, \dots, r\} \rightarrow \{1, \dots, n\}$  satisfying the following properties:*

1. *For  $1 \leq i \leq r$ ,  $m_{i,f(i)} \geq 1$ ,  $m_{ij} = 0$  if  $j < f(i)$  and  $0 \leq m_{i,f(k)} < m_{k,f(k)}$  if  $i < k$ .*
2. *The last  $m - r$  rows of  $M$  are equal to 0.*

**THEOREM 1** ([6] Theorem 2.4.3). *Let  $A$  be an  $m \times n$  matrix with coefficients in  $\mathbb{Z}$ . Then there exists a unique  $m \times n$  matrix  $B = (b_{ij})$  in HNF of the form  $B = U \cdot A$  where  $U \in \text{GL}(m, \mathbb{Z})$ .*

We will refer to matrix  $B$  as the HNF of matrix  $A$ . The construction of  $B$  and  $U$  is effective, see [6, Algorithm 2.4.4], based on Eulid's algorithm for greatest common divisor. In the following two applications of this algorithm will be considered: for computing a fan of a given wps (see Prop. 5) and the so-called  $Q$ -canonical fan of  $\mathbb{P}(Q)$  (see Prop. 6). At this purpose, a key theoretical tool is the following (for the proof see [6, §2.4.3])

**PROPOSITION 1.**

1. *Let  $L$  be a subgroup of  $\mathbb{Z}^n$ ,  $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$  a set of generators, and let  $A$  be the  $m \times n$  matrix having  $\mathbf{v}_1, \dots, \mathbf{v}_m$  as rows. Let  $B$  be the HNF of  $A$ . Then the nonzero rows of  $B$  are a basis of  $L$ .*

2. Let  $A$  be a  $m \times n$  matrix, and let  $B = U \cdot A^T$  be the HNF of the transposed of  $A$ , and let  $r$  be the number of nonzero rows of  $B$ . Then a  $\mathbb{Z}$ -basis for the kernel of  $A$  is given by the last  $m - r$  rows of  $U$ .

### 1.3. Transversion of a matrix

In the following, given a matrix  $A \in \mathrm{GL}(n, \mathbb{Q})$ , the matrix obtained by taking the transposed matrix of the inverse matrix

$$A^* := ((A)^{-1})^T$$

is called the *transverse matrix* of  $A$ . We will see in the following (see subsection 3.1, in particular Thm. 4) that *transversion* of a matrix describes, up to the multiplication by a diagonal matrix of weights, the passage from a fan to a polytope (and back) associated with the same weighted projective space  $\mathbb{P}(Q)$ .

Here are some elementary properties of transversion:

PROPOSITION 2. *Let  $A$  and  $B$  be matrices of  $\mathrm{GL}(n, \mathbb{Q})$ . Then:*

1.  $(A^*)^* = A$  i.e. *transversion is an involution in  $\mathrm{GL}(n, \mathbb{Q})$ ,*
2.  $(A \cdot B)^* = A^* \cdot B^*$ ,
3.  $\det(A^*) = 1/\det(A)$ ,
4. *if  $A$  is a upper (lower) triangular matrix then  $A^*$  is an lower (upper) triangular matrix,*
5. *if  $A \in \mathrm{GL}(n, \mathbb{Z})$  then  $A^* \in \mathrm{GL}(n, \mathbb{Z})$  too.*

### 1.4. Weighted projective spaces

In the present subsection we will briefly recall the definition and some well known fact about *weighted projective spaces* (wps in the following). Proofs and details can be recovered in the extensive treatments [12] [16], [13] and [3].

DEFINITION 2. *Set  $Q := (q_0, \dots, q_n) \in (\mathbb{N} \setminus \{0\})^{n+1}$  and consider the multiplicative group  $\mu_Q := \mu_{q_0} \oplus \dots \oplus \mu_{q_n}$  where  $\mu_{q_i}$  is the group of  $q_i$ -th roots of unity. Consider the following action of  $\mu_Q$  over the  $n$ -dimensional complex projective space  $\mathbb{P}^n$*

$$\begin{aligned} \mu_Q : \quad \mu_Q \times \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ ((\xi_j), [z_j]) &\longmapsto [\xi_j z_j] . \end{aligned}$$

*Let  $\Delta_Q \subset \mu_Q$  be the diagonal subgroup and consider the quotient group  $\mathbb{W}_Q := \mu_Q / \Delta_Q$ . Then the induced quotient space*

$$\mathbb{P}(Q) := \mathbb{P}^n / \mathbb{W}_Q$$

*is called the  $Q$ -weighted projective space ( $Q$ -wps).*



REMARK 1. If  $q$  is the greatest common divisor of  $(q_0, \dots, q_n)$  then

$$\Delta_Q \cong \mu_q$$

Therefore we get the canonical isomorphism

$$\mathbb{P}(Q) \cong \mathbb{P}\left(\frac{q_0}{q}, \dots, \frac{q_n}{q}\right)$$

For this reason in the following we will always assume that

$$q = \gcd(q_0, \dots, q_n) = 1.$$

Let us recall the following standard notation

$$(5) \quad \begin{aligned} d_j &:= \gcd(q_0, \dots, q_{j-1}, q_{j+1}, \dots, q_n), \\ a_j &:= \text{lcm}(d_0, \dots, d_{j-1}, d_{j+1}, \dots, d_n), \\ a &:= \text{lcm}(a_0, \dots, a_n). \end{aligned}$$

DEFINITION 3 (Weight vector). *In the following a weight vector  $Q = (q_0, \dots, q_n)$  will denote a  $n+1$ -tuple of coprime positive integer numbers. Referring to notation defined in (5), a weight vector  $Q$  will be called reduced if  $d_j = 1$ , or equivalently  $a_j = 1$ , for any  $j = 0, \dots, n$ .*

REMARK 2. *Every weighted projective space is a toric variety.* In fact the natural torus action over  $\mathbb{P}^n$  passes through the quotient as follows

$$\begin{array}{ccc} (\mathbb{C}^*)^n \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ \tau_Q \times \pi_Q \downarrow & & \downarrow \pi_Q \\ (\mathbb{C}^*)^n \times \mathbb{P}^n(Q) & \longrightarrow & \mathbb{P}^n(Q) \end{array}$$

where  $\pi_Q$  is the natural quotient map and  $\tau_Q$  is the quotient map associated with the action

$$\begin{aligned} \mu_Q \times (\mathbb{C}^*)^n &\longrightarrow (\mathbb{C}^*)^n \\ ((\zeta_j), (t_i)) &\longmapsto (\zeta_0^{-1} \zeta_i t_i) \end{aligned}$$

Then the torus  $(\mathbb{C}^*)^n$  can be embedded in  $\mathbb{P}(Q)$  via the following map

$$\begin{aligned} (\mathbb{C}^*)^n &\hookrightarrow \mathbb{P}(Q) \\ (t_1, \dots, t_n) &\longmapsto [1 : t_1 : \dots : t_n] \end{aligned}$$

whose image is the open subset  $\mathbb{P}(Q) \setminus \mathcal{V}(\prod_j z_j)$ .

PROPOSITION 3. *Since  $\gcd(q_0, \dots, q_n) = 1$ , the following facts are true:*

1.  $\gcd(q_j, d_j) = 1$ ,

2. if  $i \neq j$  then  $\gcd(d_i, d_j) = 1$ ,
3.  $a_j \mid q_j$ ,
4.  $\gcd(a_j, d_j) = 1$ ,
5.  $a_j d_j = a$ ,
6. setting  $q'_j := q_j / a_j$ , then  $Q' = (q'_0, \dots, q'_n)$  is reduced;  $Q'$  is then called the reduction of  $Q$ .

The proofs of these well known properties (see [13] 1.3.1) are elementary.

Instead we will prove the following property, which does not appear in the main treatments of the subject.

PROPOSITION 4. Let  $Q = (q_0, \dots, q_n)$  be a weight vector and  $Q' = (q'_0, \dots, q'_n)$  be its reduction. Define

$$(6) \quad \delta := \text{lcm}(q_0, \dots, q_n) \quad \text{and} \quad \delta' := \text{lcm}(q'_0, \dots, q'_n).$$

Then  $\delta = a\delta'$ , where  $a$  is defined in (5).

REMARK 3. The Proposition 4 still holds when  $q := \gcd(q_0, \dots, q_n) > 1$ .

*Proof of Proposition 4.* We will prove that  $\delta$  divides  $a\delta'$  and, viceversa,  $a\delta'$  divides  $\delta$ .

On the one hand, to show that  $\delta$  divides  $a\delta'$  it suffices to show that  $q_j$  divides  $a\delta'$  for every  $j$  and this fact follows immediately by definitions since  $q_j = a_j q'_j \mid a\delta'$ .

On the other hand, by definitions of  $a$  and  $\delta'$ , to prove that  $a\delta'$  divides  $\delta$  it suffices to prove that  $a_i q'_k \mid \delta$ , which is  $a_i \mid \frac{\delta}{q'_k} a_k$ , for every  $i, k$ . By the definition of  $a_i$  given in (5), the latter is obtained by showing that  $d_j$  divides  $\frac{\delta}{q'_k} a_k$  for every  $j, k$ .

If  $j \neq k$  then  $d_j$  divides  $a_k$  and we are done.

Suppose now  $j = k$ . Let  $p$  be a prime dividing  $d_k$  and let  $p^t, p^r$  be the highest powers of  $p$  dividing  $d_k$  and  $q_k$  respectively. Then  $p^t$  divides  $q_i$  for every  $i \neq k$ , by the definition of  $d_k$ : in particular  $p^t \mid \delta$ . If  $r \geq t$  then

$$\forall i \quad p^t \mid q_i \Rightarrow \forall j \quad p^t \mid d_j \Rightarrow \forall k \quad p^t \mid a_k.$$

If  $r < t$  then  $p^{t-r}$  divides  $\frac{\delta}{q'_k}$ ; moreover  $p^r$  divides  $d_i$  for every  $i \neq k$ , since  $p^r \mid q_k$  and  $p^t \mid q_i$  for every  $i \neq k$ : then  $p^r$  divides  $a_k$ . Therefore  $p^t$  divides  $\frac{\delta}{q'_k} a_k$ . Thus we proved that  $d_k$  divides  $\frac{\delta}{q'_k} a_k$ .  $\square$

Let us now recall the following well-known result to which we will refer below as to the *Reduction Theorem*.

THEOREM 2 (Reduction Theorem ([12] §1, [13] 1.3.1)). Let  $Q' = (q'_0, \dots, q'_n)$  be the reduced weight vector of  $Q = (q_0, \dots, q_n)$ . Then

$$(7) \quad \mathbb{P}(Q) \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^* = \mathbb{P}(Q')$$

where the quotient is realized by means of the (reduced) action

$$\begin{aligned} \mathbf{v}_{Q'} : \mathbb{C}^* \times \mathbb{C}^{n+1} &\longrightarrow \mathbb{C}^{n+1} \\ (t, (z_j)) &\longmapsto (t^{q'_0} z_0, \dots, t^{q'_n} z_n) . \end{aligned}$$

Let us end up this preliminary section with the following technical statement which will be useful below. Partial proofs of this result may recovered from [4] §2 and [14] Prop. 2.3. Moreover it is certainly well-know to experts. However for purposes of definiteness we include here a detailed proof.

LEMMA 1. *Let  $Q = (q_0, \dots, q_n)$  be a weight vector; let  $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$  be a set of vectors in  $\mathbb{Q}^n$ , generating  $\mathbb{Q}^n$  and such that  $\sum_{j=0}^n q_j \mathbf{v}_j = 0$ . Let  $L$  be the lattice generated in  $\mathbb{Q}^n$  by  $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$  and  $L'$  be the sublattice generated by  $\{q_0 \mathbf{v}_0, \dots, q_n \mathbf{v}_n\}$ . Then the following properties hold:*

- (a)  $[L : L'] = \prod_{j=0}^n q_j$ ;
- (b) let  $V := (v_{ij})$  be the  $n \times (n+1)$  matrix whose columns are given by components of  $\mathbf{v}_0, \dots, \mathbf{v}_n$  over a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of  $L$  i.e.  $\mathbf{v}_j = \sum_{i=1}^n v_{ij} \mathbf{e}_i$ , for every  $j = 0, \dots, n$ , and denote by  $V_j$  the  $n$ -minor of  $V$  obtained by deleting the  $j$ -th column as in (3). Then
 
$$\forall j = 0, \dots, n \quad V_j = (-1)^{\varepsilon+j} q_j, \text{ for a fixed } \varepsilon \in \{0, 1\},$$
- (c)  $\forall j = 0, \dots, n \quad \mathbf{v}_j = d_j \mathbf{n}_j$ , where  $\mathbf{n}_j$  is the generator of the semigroup  $\langle \mathbf{v}_j \rangle \cap L$  and  $d_j$  is defined in (5); in particular  $L$  is the lattice generated by  $\{\mathbf{n}_0, \dots, \mathbf{n}_n\}$ ; moreover  $\{\mathbf{n}_0, \dots, \mathbf{n}_n\}$  satisfy the hypotheses of this Lemma with respect to the reduced weight vector  $Q'$  i.e. they generate  $\mathbb{Q}^n$  and  $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$ .

*Proof.* For (a), observe that  $L'$  has  $q_1 \mathbf{v}_1, \dots, q_n \mathbf{v}_n$  as a basis. Then  $L'$  has index  $\prod_{j=1}^n q_j$  in the lattice  $L_0$  generated by  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . The quotient  $L/L_0$  is cyclic generated by the image of  $\mathbf{v}_0$ , so that  $[L : L_0]$  divides  $q_0$ . If  $r \mathbf{v}_0 \in L_0$ , with  $r \in \mathbb{Z}$  then  $r \mathbf{v}_0 = \sum_{j=1}^n s_j \mathbf{v}_j$  with  $s_1, \dots, s_n \in \mathbb{Z}$ . Since  $\gcd(q_0, \dots, q_n) = 1$  then there exists  $\lambda \in \mathbb{Z}$  such that  $r = -\lambda q_0$ ,  $s_i = \lambda q_i$  for  $i = 1, \dots, n$ ; in particular  $q_0$  divides  $r$ , so that  $[L : L_0] = q_0$  and  $[L : L'] = [L : L_0][L_0 : L'] = \prod_{i=0}^n q_i$ .

(b): for  $j = 0, \dots, n$ , let  $L_j$  be the lattice generated by  $\mathbf{v}_0, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_n$ . Then  $|V_j| = [L : L_j] = q_j$ , as we have shown in (a) for the case  $j = 0$ . Let  $\varepsilon \in \{0, 1\}$  be such that  $V_0 = (-1)^\varepsilon q_0$ . Then

$$\forall j = 0, \dots, n \quad V_j = (-1)^j \frac{q_j}{q_0} V_0 = (-1)^{\varepsilon+j} q_j$$

since  $\sum_{j=0}^n q_j \mathbf{v}_j = 0$ .

(c): we have

$$\forall j = 0, \dots, n \quad q_j \mathbf{v}_j = - \sum_{k \neq j} q_k \mathbf{v}_k = -d_j \sum_{k \neq j} \tilde{q}_k \mathbf{v}_k$$

where  $\tilde{q}_k := q_k/d_j \in \mathbb{N}$ . By (1) in Proposition 3,  $\gcd(q_j, d_j) = 1$  meaning that

$$\forall j = 0, \dots, n \quad \exists \mathbf{v}'_j \in L : \mathbf{v}_j = d_j \mathbf{v}'_j .$$

Then (5) in Proposition 3 allows to write

$$(8) \quad 0 = \sum_{j=0}^n q_j \mathbf{v}_j = \sum_{j=0}^n (q'_j a_j) (d_j \mathbf{v}'_j) = a \sum_{j=0}^n q'_j \mathbf{v}'_j \implies \sum_{j=0}^n q'_j \mathbf{v}'_j = 0.$$

Moreover  $\mathbf{v}'_0, \dots, \mathbf{v}'_n$  generate  $L$  and (a) ensures that the following index

$$(9) \quad [L : \langle q'_j \mathbf{v}'_j \mid j = 0, \dots, n \rangle] = \prod_{j=0}^n q'_j.$$

Then the proof ends up by showing that, for all  $j$ ,  $\mathbf{v}'_j = \mathbf{n}_j$ . With this goal in mind, consider  $h_j \in \mathbb{N}$  such that  $\mathbf{v}'_j = h_j \mathbf{n}_j$ . If  $V' = (v'_{ij})$  is the matrix of components of  $\mathbf{v}'_0, \dots, \mathbf{v}'_n$ , over the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of  $L$ , then

$$\forall j = 0, \dots, n \quad |V'_j| = q'_j$$

On the other hand

$$\mathbf{v}'_0 \in h_0 L \implies \forall i = 1, \dots, n \quad h_0 \mid v'_{i0} \implies \forall k = 1, \dots, n \quad h_0 \mid |V'_k| = q'_k.$$

Therefore (6) in Proposition 3 implies that

$$h_0 \mid \gcd(q'_1, \dots, q'_n) = 1 \implies h_0 = 1$$

Analogously  $h_j = 1$ , for all  $1 \leq j \leq n$ . Hence  $\mathbf{v}'_j = \mathbf{n}_j$ . □

## 2. Characterization of fans giving $\mathbb{P}(Q)$

### 2.1. Characterizing the fan

Let us fix an  $n$ -dimensional lattice  $N$  and a subset of  $n+1$  vectors  $\{\mathbf{v}_0, \dots, \mathbf{v}_n\} \subset N$ . The following theorem is an application of the previous Lemma 1 to known results such as e.g. Lemma 2.11 in [2], Prop. 5.4 in [5] and Thm. 3.6 in [7].

**THEOREM 3.** *Let  $Q = (q_0, \dots, q_n)$  be a weight vector. Consider the fan  $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$  and the associated matrix  $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$  with respect to a fixed basis of  $N$ . Then the following facts are equivalent:*

1.  $\Sigma$  is a fan of  $\mathbb{P}(Q)$ ,
2.  $\sum_{j=0}^n q_j \mathbf{v}_j = 0$  and the sub-lattice  $N' := \langle q_0 \mathbf{v}_0, \dots, q_n \mathbf{v}_n \rangle \subset N$  has finite index

$$[N : N'] = \prod_{j=0}^n q_j,$$

3.  $\forall j = 0, \dots, n \quad V_j = (-1)^{\varepsilon+j} q_j$ , for a fixed  $\varepsilon \in \{0, 1\}$ ,

4.  $q_0 \mathbf{v}_0 = -\sum_{i=1}^n q_i \mathbf{v}_i$  and  $|V_0| := |\det(\mathbf{v}_1, \dots, \mathbf{v}_n)| = q_0$ .

*Proof of Theorem 3.* (1)  $\Rightarrow$  (2). A fan of the wps  $\mathbb{P}(Q)$  with  $Q = (q_0, \dots, q_n)$  is presented in [15] at the end of §2.3. Then, by Lemma 1(a) one may check that fan to satisfy conditions stated in (2).

(2)  $\Rightarrow$  (3). This is Lemma 1(b).

(3)  $\Rightarrow$  (4). For any  $k = 1, \dots, n$  consider the  $(n+1) \times (n+1)$  matrix

$$A_k := \begin{pmatrix} v_{k0} & \cdots & v_{kn} \\ \vdots & & \vdots \\ v_{n0} & \cdots & v_{nn} \end{pmatrix}.$$

Since the first and the  $(k+1)$ -st rows of  $A_k$  are equal we get

$$\forall k = 1, \dots, n \quad 0 = \det(A_k) = \sum_{j=0}^n (-1)^j v_{kj} V_j \stackrel{(3)}{=} (-1)^e \sum_{j=0}^n q_j v_{kj} \Rightarrow \sum_{j=0}^n q_j \mathbf{v}_j = 0.$$

(4)  $\Rightarrow$  (2). Since  $|V_0| = q_0$  then  $\{q_1 \mathbf{v}_1, \dots, q_n \mathbf{v}_n\}$  is a basis of the sub-lattice  $N'$ . Hence

$$[N : N'] = |\det(q_1 \mathbf{v}_1, \dots, q_n \mathbf{v}_n)| = \left( \prod_{i=1}^n q_i \right) |V_0| \stackrel{(4)}{=} \prod_{j=0}^n q_j.$$

(2)  $\Rightarrow$  (1). First note that (2) and Lemma 1(a) imply that  $\mathbf{v}_0, \dots, \mathbf{v}_n$  generate  $N$ . It follows by Lemma 1(c) that  $\mathbf{n}_0, \dots, \mathbf{n}_n$  generate  $N$  and  $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$ .

Let  $D_j$  be the torus invariant divisor associated with the 1-dimensional cone  $\langle \mathbf{n}_j \rangle \in \Sigma(1)$  for the toric variety with fan  $\Sigma$ . Consider the sequence

$$(10) \quad 0 \longrightarrow M \xrightarrow{\text{div}} \bigoplus_{j=0}^n \mathbb{Z} \cdot D_j \xrightarrow{d} \mathbb{Z} \longrightarrow 0$$

where  $\text{div}(\mathbf{m}) = \sum_{j=0}^n \langle \mathbf{m}, \mathbf{n}_j \rangle D_j$  and  $d(\sum_{j=0}^n b_j D_j) = \sum_{j=0}^n b_j q'_j$ . Then  $\text{div}$  is injective since the  $\mathbf{n}_j$  span  $N = M^\vee$  and  $d$  is surjective since  $\gcd(q'_0, \dots, q'_n) = 1$ . Furthermore,  $d \circ \text{div} = 0$  follows easily from  $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$ .

Hence, to prove that (10) is exact, it suffices to show that  $\ker(d) \subset \text{Im}(\text{div})$ . Take  $\sum_{j=0}^n b_j D_j \in \ker(d)$ . Since  $\mathbf{n}_1, \dots, \mathbf{n}_n$  are linearly independent over  $\mathbb{Q}$ , one can find  $\mathbf{m} \in M \otimes \mathbb{Q}$  such that  $\langle \mathbf{m}, \mathbf{n}_i \rangle = b_i$  for  $1 \leq i \leq n$ . Since  $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$  and  $\sum_{j=0}^n b_j q'_j = 0$ , we obtain

$$q'_0 \langle \mathbf{m}, \mathbf{n}_0 \rangle = -\sum_{i=1}^n \langle \mathbf{m}, q'_i \mathbf{n}_i \rangle = -\sum_{i=1}^n q'_i \langle \mathbf{m}, \mathbf{n}_i \rangle = -\sum_{i=1}^n q'_i b_i = q'_0 b_0.$$

It follows that  $\langle \mathbf{m}, \mathbf{n}_0 \rangle = b_0$ . Thus  $\langle \mathbf{m}, \mathbf{n}_j \rangle = b_j \in \mathbb{Z}$  for all  $j$ . This implies  $\mathbf{m} \in M$  since  $\mathbf{n}_0, \dots, \mathbf{n}_n$  span  $N = M^\vee$ , and the desired exactness follows immediately.

We are now in a position to apply the Cox Theorem, [8] Thm. 2.1, to give a geometric quotient description of  $X(\Sigma)$ . In particular the exact sequence (10) suffices

to show that the *Chow group* of Weil divisors modulo rational equivalence for the toric variety  $X(\Sigma)$  is given by  $A_{n-1}(X) \cong \mathbb{Z}$  (see e.g. [15] §3.4). Let  $S = \mathbb{C}[x_0, \dots, x_n]$  be the polynomial ring obtained by associating the variable  $x_j$  with the 1-dimensional cone  $\langle \mathbf{n}_j \rangle \in \Sigma(1)$ . The grading on  $S$  is defined by setting

$$\deg(x_j) := \deg(d(D_j)) = q_j.$$

Since  $\text{Hom}(A_{n-1}(X), \mathbb{C}^*) \cong \mathbb{C}^*$ , it is then possible to exhibit  $X(\Sigma)$  as a geometric quotient

$$X(\Sigma) \cong (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*$$

where the quotient is realized by means of the action  $\mathbf{v}_{Q'}$  in the statement of the Reduction Theorem 2. Then  $X(\Sigma) \cong \mathbb{P}(Q') \cong \mathbb{P}(Q)$ .  $\square$

The following definition will be useful in section 3, when speaking about the *fan-polytope correspondence*:

**DEFINITION 4** (*F*-admissible matrices). *A matrix  $V \in \text{Mat}(n, n+1, \mathbb{Z})$  will be called *F*-admissible if it satisfies the following conditions*

1. *the matrix  $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$  admits only nonvanishing coprime maximal minors i.e.  $\forall j = 0, 1, \dots, n \quad V_j \neq 0$  and  $\gcd(V_j \mid 0 \leq j \leq n) = 1$ ;*
2. *the columns  $\mathbf{v}_j$  of  $V$  satisfy one of the equivalent conditions (2), (3), (4) of Theorem 3 with respect to the weights  $q_j := |V_j|$ .*

*The subset of *F*-admissible matrices will be denoted by  $\mathfrak{V}_n \subset \text{Mat}(n, n+1, \mathbb{Z})$ .*

## 2.2. Hermite normal form of weights and fans of $\mathbb{P}(Q)$

The following result, which is a direct consequence of Theorem 3, exhibit a rather surprising method to get a fan of a given wps  $\mathbb{P}(Q)$ : in fact this fan turns out to be *encoded* in the switching matrix giving the HNF of the transposed weight vector  $Q^T$ . Since such a matrix is obtained by a well known algorithm, based on Euclid's algorithm for greatest common divisor, [6] Algorithm 2.4.4, this gives a constructive method to produce a fan of  $\mathbb{P}(Q)$  which can be performed by any procedures computing elementary linear algebra operations.

**PROPOSITION 5.** *Let  $Q = (q_0, \dots, q_n)$  be a weight vector,  $B$  the HNF of the transposed vector  $Q^T$  and  $U \in \text{GL}(n+1, \mathbb{Z})$  be such that  $U \cdot Q^T = B$ . Let  $C$  be the matrix consisting of the last  $n$  rows of  $U$  and let  $\mathbf{v}_j$  be the  $j^{\text{th}}$  column vector of  $C$ , for  $0 \leq j \leq n$ . Let  $L, L'$  be the lattices generated in  $\mathbb{Z}^n$  by  $\mathbf{v}_0, \dots, \mathbf{v}_n$  and  $q_0\mathbf{v}_0, \dots, q_n\mathbf{v}_n$  respectively. Then*

$$1. \ B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

2.  $L = \mathbb{Z}^n$ ;
3.  $\sum_{j=0}^n q_j \mathbf{v}_j = 0$ ;
4. there exists  $\varepsilon \in \{0, 1\}$  such that  $C_j = (-1)^{\varepsilon+j} q_j$  for all  $0 \leq j \leq n$ .
5.  $[L : L'] = \prod_{j=0}^n q_j$ .

As a consequence of Theorem 3,  $\text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$  is a fan of  $\mathbb{P}(Q)$ .

*Proof.* The rank of  $Q$  is 1, so by definition of HNF,  $B = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$  with  $a \geq 1$ . By the

equality  $Q^T = U^{-1} \cdot B$  we see that  $a$  must divide  $q_0, \dots, q_n$ , so that  $a = 1$ ; this proves (1). Then (3) follows immediately by  $U \cdot Q^T = B$ .

To prove part (2), let  $(c_0, \dots, c_n)$  be the 2nd column of  $U^{-1}$ . Then  $U \cdot U^{-1} = I$  easily implies that  $\sum_{j=0}^n c_j \mathbf{v}_j = \mathbf{e}_1$ , the first standard basis vector of  $\mathbb{Z}^n$ . Columns 3,  $\dots$ ,  $n+1$  of  $U^{-1}$  similarly show that  $\mathbf{e}_2, \dots, \mathbf{e}_n$  are in the sublattice generated by  $\mathbf{v}_0, \dots, \mathbf{v}_n$ . Hence this sublattice must be  $\mathbb{Z}^n$ .

Finally parts (4) and (5) follow immediately from parts (2) and (3) of Lemma 1.  $\square$

### 2.3. A $Q$ -canonical fan of $\mathbb{P}(Q)$

In the present subsection we want to use the characterization (4) in Theorem 3 to get a  $Q$ -canonical fan of  $\mathbb{P}(Q)$ , in the sense that the associated fan matrix is in HNF, up to a permutation of columns (see the following Remark 4). This fact presents the fan in a triangular shape and generated by as many as possible of the vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  in a given basis of the lattice  $N$ . Moreover it turns out to be a convenient procedure to get a fan of  $\mathbb{P}(Q)$  by hands (see Example 1 below).

**PROPOSITION 6.** *Let  $Q = (q_0, \dots, q_n)$  be a weight vector. For any  $j$  with  $1 \leq j \leq n$ , define  $k_j := \gcd(q_0, q_j, q_{j+1}, \dots, q_n)$ . Then:*

1.  $k_j \mid k_{j+1}$ ,
2. either  $k_n = (q_0, q_n) = 1$  or there exists a positive integer  $i$ , with  $1 \leq i \leq n-1$ , such that  $k_i = 1$  and  $k_{i+1} > 1$ ,
3. consider a upper triangular matrix  $V^0 = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \text{Mat}(n, n, \mathbb{Z})$  whose columns  $\mathbf{v}_j$  are such that:

$$\begin{aligned} \forall 1 \leq j \leq i-1 \quad \mathbf{v}_j &= \mathbf{e}_j \\ \forall i \leq j \leq n-1 \quad v_{jj} &= k_{j+1}/k_j \\ v_{nn} &= q_0/k_n \end{aligned}$$

where  $v_{kj}$  is the  $k$ -th entry of the column  $\mathbf{v}_j$ ; then there exists a choice for  $v_{kj}$  with  $i \leq j$  and  $k < j$  such that  $V^0$  can be completed to a matrix  $V = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n) \in \text{Mat}(n, n+1, \mathbb{Z})$  whose columns satisfy the following condition

$$\sum_{j=0}^n q_j \mathbf{v}_j = 0 ;$$

in particular the columns of  $V$  satisfy condition (4) of Theorem 3, hence generate a fan of  $\mathbb{P}(Q)$ .

4. there exists a unique choice of the previous matrices  $V$  and  $V^0$  such that  $V^0$  is in HNF with only nonnegative entries; then the column  $\mathbf{v}_0$  in  $V$  admits only negative entries. Moreover the matrix  $V' = (\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_0)$  is in HNF.

*Proof.* (1) is obvious. (2) follows from (1) by recalling the hypothesis

$$k_1 = \gcd(q_0, q_1, \dots, q_n) = 1 .$$

To prove (3) we have to show the existence of an integral vector  $\mathbf{v}_0 = \begin{pmatrix} v_{10} \\ \vdots \\ v_{n0} \end{pmatrix}$  and integers  $v_{jk}$  satisfying the following equations

$$(11) \quad \forall 1 \leq j \leq i-1 \quad q_0 v_{j0} + q_j + \sum_{k=i}^n q_k v_{jk} = 0$$

$$(12) \quad \forall i \leq j \leq n-1 \quad q_0 v_{j0} + q_j \frac{k_{j+1}}{k_j} + \sum_{k=j+1}^n q_k v_{jk} = 0$$

$$(13) \quad q_0 v_{n0} + q_n \frac{q_0}{k_n} = 0 .$$

The last equation (13) is clearly satisfied by putting  $v_{n0} = -q_n/k_n = -q_n/(q_0, q_n)$ . The  $j$ -th equation in (12) admits integer solutions for  $(v_{j0}, v_{j,j+1}, \dots, v_{jn})$  if and only if

$$\gcd(q_0, q_{j+1}, \dots, q_n) = k_{j+1} \mid q_j \frac{k_{j+1}}{k_j}$$

which is clearly true since  $k_j \mid q_j$ , by definition. Finally the  $j$ -th equation in (11) admits integer solutions for  $(v_{j0}, v_{ji}, v_{j,i+1}, \dots, v_{jn})$  if and only if

$$\forall 1 \leq j \leq i-1 \quad \gcd(q_0, q_i, \dots, q_n) = k_i \mid q_j$$

which is clearly true since  $k_i = 1$ , by the previous part (2). Recall now that  $V^0$  is a triangular matrix, giving

$$\det(V^0) = \prod_{j=1}^n v_{jj} = k_{i+1} \cdot \frac{k_{i+2}}{k_{i+1}} \cdots \frac{k_n}{k_{n-1}} \cdot \frac{q_0}{k_n} = q_0$$



which is enough to get condition (4) of Theorem 3 for the columns of  $V$ .

To prove (4) let us first of all observe that, for any  $1 \leq j \leq i-1$ ,  $\mathbf{v}_j = \mathbf{e}_j$ , meaning that the first  $i-1$  columns of  $V^0$  are composed of nonnegative entries satisfying the HNF conditions. Moreover  $V^0$  is upper triangular. Then it remains to prove that there exists a unique choice for  $v_{jk}$  such that

$$\forall k : i \leq k \leq n, \forall j : j < k \quad 0 \leq v_{jk} < v_{kk}.$$

The  $j$ -th equation in (12) can be rewritten as follows

$$q_0 v_{j0} + q_n v_{jn} = -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-1} q_k v_{jk}.$$

Fixing variables  $v_{jk}$ , for  $j+1 \leq k \leq n-1$ , the previous diophantine equation admits solutions for  $v_{j0}, v_{jn}$  if and only if

$$(14) \quad k_n = \gcd(q_0, q_n) \mid -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-1} q_k v_{jk}.$$

Moreover, given a particular solution  $v_{jn}^{(0)}$ , all the possible integer solutions for  $v_{jn}$  are given by

$$v_{jn} = v_{jn}^{(0)} - \frac{q_0}{k_n} \cdot h_{jn} = v_{jn}^{(0)} - v_{nn} \cdot h_{jn}, \quad \forall h_{jn} \in \mathbb{Z}.$$

Divide  $v_{jn}^{(0)}$  by  $v_{nn}$ . Then the remainder of such a division gives a unique choice for  $v_{jn}$  such that

$$\forall i \leq j \leq n-1 \quad 0 \leq v_{jn} < v_{nn}.$$

Analogously the  $j$ -th equation in (11) can be rewritten as follows

$$q_0 v_{j0} + q_n v_{jn} = -q_j - \sum_{k=i}^{n-1} q_k v_{jk}$$

and the same argument ensures the existence of a unique choice for  $v_{jn}$  such that

$$\forall 1 \leq j \leq i-1 \quad 0 \leq v_{jn} < v_{nn}.$$

Then the last column in  $V^0$  can be uniquely chosen with non-negative entries satisfying the HNF condition. Iteratively, condition (14) is satisfied if and only if there exist integer solutions for  $x, v_{jk}$  in the diophantine equation

$$k_n x + q_{n-1} v_{j,n-1} = -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-2} q_k v_{jk}$$

which is if and only if

$$\gcd(k_n, q_{n-1}) = \gcd(q_0, q_{n-1}, q_n) =: k_{n-1} \mid -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-2} q_k v_{jk}.$$

In particular, given a solution  $v_{j,n-1}^{(0)}$ , all the possible integer solutions for  $v_{j,n-1}$  are given by

$$v_{j,n-1} = v_{j,n-1}^{(0)} - \frac{k_n}{k_{n-1}} \cdot h_{j,n-1} = v_{j,n-1}^{(0)} - v_{n-1,n-1} \cdot h_{j,n-1}, \quad \forall h_{j,n-1} \in \mathbb{Z}.$$

Therefore, the division algorithm ensures the existence of a unique choice for  $v_{j,n-1}$  such that

$$\forall i \leq j \leq n-1 \quad 0 \leq v_{j,n-1} < v_{n-1,n-1}.$$

The same argument ensures the existence of a unique choice for  $v_{j,n-1}$  such that

$$\forall 1 \leq j \leq i-1 \quad 0 \leq v_{j,n-1} < v_{n-1,n-1}.$$

Then the  $(n-1)$ -st column in  $V^0$  can be uniquely chosen with non-negative entries satisfying the HNF condition. By completing the iteration,  $V_0$  can then be uniquely chosen in HNF. Consequently  $\mathbf{v}_0$  has to necessarily admit only negative entries. To prove that  $V'$  is in HNF it suffices to observe that, for  $V'$ , the function  $f: \{1, \dots, n\} \rightarrow \{1, \dots, n+1\}$ , in Definition 1, is given by setting  $f(i) = i$ , for any  $1 \leq i \leq n$ . Then  $V'$  is in HNF if and only if  $V^0$  is in HNF, since there are no condition for the entries of  $\mathbf{v}_0$  which is the  $(n+1)$ -st column of  $V'$ .  $\square$

REMARK 4. When the weight vector  $Q$  is fixed, a significant consequence of Proposition 6 is that the fan of  $\mathbb{P}(Q)$  presented in (4) is unique and is given by the HNF of a matrix  $V$  associated with any fan of  $\mathbb{P}(Q)$ . Clearly the uniqueness of the  $Q$ -canonical fan of  $\mathbb{P}(Q)$  depends on the weights order in  $Q$ . Then we can't define a *canonical* fan of  $\mathbb{P}(Q)$  but just a  $Q$ -canonical one.

EXAMPLE 1. Let us apply the Proposition 6 to produce by hand the  $Q$ -canonical fan (hence a fan) of  $\mathbb{P}(Q)$  for  $Q = (2, 3, 4, 15, 25)$ . First of all observe that in this case

$$k_1 = \gcd(Q) = 1, \quad k_2 = d_1 = 1, \quad k_3 = \gcd(2, 15, 25) = 1, \quad k_4 = \gcd(2, 25) = 1.$$

The matrix  $V'$  in Proposition 6(4) is in HNF, then it looks as follows

$$V' = \begin{pmatrix} 1 & 0 & 0 & v_{1,3} & v_{1,0} \\ 0 & 1 & 0 & v_{2,3} & v_{2,0} \\ 0 & 0 & 1 & v_{3,3} & v_{3,0} \\ 0 & 0 & 0 & 2 & v_{4,0} \end{pmatrix}$$

with  $0 \leq v_{k,3} \leq 1$ , for  $1 \leq k \leq 3$ . Moreover we get the following conditions

$$\begin{aligned} 0 &= q_0 v_{4,0} + \frac{q_0 q_4}{k_4} = 2v_{4,0} + 50 \Rightarrow v_{4,0} = -25 \\ 0 &= q_0 v_{3,0} + \frac{k_3 q_3}{k_4} + q_4 v_{3,3} = 2v_{3,0} + 15 + 25v_{3,3} \Rightarrow v_{3,3} = 1 \text{ and } v_{3,0} = -20 \\ 0 &= q_0 v_{2,0} + \frac{k_2 q_2}{k_3} + q_4 v_{2,3} = 2v_{2,0} + 4 + 25v_{2,3} \Rightarrow v_{2,3} = 0 \text{ and } v_{2,0} = -2 \\ 0 &= q_0 v_{1,0} + \frac{k_1 q_1}{k_2} + q_4 v_{1,3} = 2v_{1,0} + 3 + 25v_{1,3} \Rightarrow v_{1,3} = 1 \text{ and } v_{1,0} = -14 \end{aligned}$$

giving the following  $Q$ -canonical fan for  $\mathbb{P}(2, 3, 4, 15, 25)$  :

$$\Sigma = \text{fan} \left( \begin{pmatrix} -14 \\ -2 \\ -20 \\ -25 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right).$$

REMARK 5. Proposition 6 above has to be compared with results in §3 and §4 of [7]. In particular parts from (1) to (3) give a rewrite of Proposition 3.2, Remark 3.3 and Theorem 3.6 in [7]. For what concerns part (4), although it apparently looks to be related with Remark 4.3 and Theorem 4.5 in [7], it seems to us to be a rather new result in the literature. In fact Conrads, in §4 of [7], discusses HNFs of square matrices in  $\text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$  since he's interested in classifying isomorphism classes of simplices of a given type. Here the aim is quite different since we study HNF of the simplex itself, which turns out to be unique and then identifying the  $Q$ -canonical fan of  $\mathbb{P}(Q)$ .

### 3. Characterization of polytopes giving $\mathbb{P}(Q)$

#### 3.1. From fans to polytopes and back

We shall use the following notation: given a  $n \times (n+1)$  matrix  $V = (\mathbf{v}_0, \dots, \mathbf{v}_n) = (v_{ij})$  with  $1 \leq i \leq n, 0 \leq j \leq n$ , the  $n \times n$  sub-matrix of  $V$  obtained by removing the first column is denoted by  $V^0 = (\mathbf{v}_1, \dots, \mathbf{v}_n) = (v_{ik})$  with  $1 \leq i \leq n$  and  $1 \leq k \leq n$ .

DEFINITION 5. Let  $V \in \text{Mat}(n, n+1, \mathbb{Z})$  be a matrix whose maximal minors do not vanish i.e., in the same notation given above,  $V_l \neq 0$  for every  $0 \leq l \leq n$ . Consider the vector of absolute values of maximal minors  $Q = (|V_0|, \dots, |V_n|)$ . Recalling 1.3, the  $(0, Q)$ -weighted transverse matrix of  $V$  (or simply weighted transverse) is defined to be the following  $n \times n$  rational matrix

$$(V^0)_Q^* := (V^0)^* \cdot (\delta I_Q^0)$$

where  $I_Q^0 := \text{diag}(1/|V_1|, \dots, 1/|V_n|)$  and  $\delta := \text{lcm}(|V_0|, \dots, |V_n|)$ .

REMARK 6. If  $V \in \mathfrak{V}_n$ , as defined in the Definition 4, then Theorem 4 below implicitly shows that the weighted transverse matrix  $(V^0)_Q^*$  is a  $n \times n$  integral matrix. In particular this fact is also proved explicitly in the following Proposition 7.

PROPOSITION 7. If  $V = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n)$  is a fan matrix of  $\mathbb{P}(Q)$ , with  $Q = (q_0, \dots, q_n)$ , then the weighted transverse  $(V^0)_Q^*$  has integral entries.

*Proof.* Recall that the adjoint matrix of an invertible square matrix  $A$  is defined by setting  $\text{Adj}(A) := \det(A) A^{-1}$ . Set  $W = \text{Adj}(V^0)$  and let  $\mathbf{w}_i$  be the  $i$ -th row of  $W$ .

Observe that parts (3) and (4) in Theorem 3 give, for  $i = 1, \dots, n$ ,

$$\begin{aligned} |\mathbf{w}_i \cdot \mathbf{v}_i| &= |\det(V^0)| = q_0 \\ \mathbf{w}_i \cdot \mathbf{v}_k &= 0 \quad \text{for } 1 \leq k \leq n \text{ and } k \neq i \\ |\mathbf{w}_i \cdot \mathbf{v}_0| &= |\det(V^0)| \frac{q_i}{q_0} = q_i, \end{aligned}$$

where the dot product is the usual matrix product. Therefore  $\frac{\delta}{q_0 q_i} \mathbf{w}_i \cdot \mathbf{v}_j \in \mathbb{Z}$  for any  $0 \leq j \leq n$ . This means that

$$\forall 1 \leq i \leq n \quad \frac{\delta}{q_0 q_i} \mathbf{w}_i \in \mathbb{Z}^n$$

since  $\mathcal{L}(\mathbf{v}_0, \dots, \mathbf{v}_n) = \mathbb{Z}^n$ . The proof concludes by transposing  $W$ .  $\square$

Let us denote by  $\mathcal{O}_{\mathbb{P}(Q)}(1)$ , or  $\mathcal{O}(1)$  for short, the generator of the Picard group  $\text{Pic}(\mathbb{P}(Q)) \cong \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}(Q)}(1)$ .

**PROPOSITION 8.** *If  $D_j$  is the torus invariant divisor associated with  $\langle \mathbf{v}_j \rangle \in \Sigma(1)$  then  $(\delta'/q'_j)D_j$  is an ample divisor in the linear system  $|\mathcal{O}_{\mathbb{P}(Q)}(1)|$ , where as usual  $Q' = (q'_0, \dots, q'_n)$  is the reduced weight vector of  $Q$  and  $\delta' = \text{lcm}(Q')$ .*

*Proof.* Recall the exact sequence (10) showing that the Chow group of  $\mathbb{P}(Q)$  is given by  $A_{n-1}(\mathbb{P}(Q)) \cong \mathbb{Z}$ . By construction, the morphism  $d : \bigoplus_{i=0}^n \mathbb{Z} \cdot D_j \rightarrow \mathbb{Z}$  sends a Weil divisor  $\sum_{j=0}^n b_j D_j$  to the generator  $1 \in \mathbb{Z}$  if and only if  $(b_0, \dots, b_n)$  is a solution of the diophantine equation  $\sum_{j=0}^n q'_j x_j = 1$ . It is a well known fact that the Picard group of a normal toric variety can be identified with the subgroup of  $A_{n-1}(X)$  generated by the classes of torus invariant Cartier divisors (see e.g. [15] § 3.4, [10] § 4.2). In particular  $\text{Pic}(\mathbb{P}(Q)) \subset A_{n-1}(\mathbb{P}(Q)) \cong \mathbb{Z}$  is a free cyclic subgroup. Then a generator of  $\text{Pic}(\mathbb{P}(Q))$  is given by a suitable multiple  $kD$  of a generator  $D$  of  $A_{n-1}(\mathbb{P}(Q))$ , where  $k$  is the least positive integer number such that  $kD$  is Cartier. There is a Criterion to determine when a Weil divisor of a toric variety is a Cartier divisor ([17] Prop. 2.4) which applied to the case of  $\mathbb{P}(Q)$  can be rewritten as follows:

$$(15) \quad \sum_{j=0}^n b_j D_j \text{ is a Cartier divisor} \iff \forall 0 \leq l \leq n \quad \exists \mathbf{u}_l \in M : \forall j \neq l \quad \langle \mathbf{u}_l, \mathbf{n}_j \rangle = b_j$$

where  $\mathbf{n}_j$  is a generator of the monoid  $\langle \mathbf{v}_j \rangle \cap N$ . Recall the exact sequence (10) and let  $D = \sum_{j=0}^n b_j D_j$  be a generator of  $A_{n-1}(\mathbb{P}(Q))$ , i.e.  $d(D) = 1$ , and consider the positive integer multiple  $kD$ . Then (15) gives that  $kD$  is a Cartier divisor if and only if, for every  $l = 0, \dots, n$ ,

$$(16) \quad \exists \mathbf{u}_l \in M : \forall j \neq l \quad \langle \mathbf{u}_l, \mathbf{n}_j \rangle = kb_j.$$

Since  $\text{div}(\mathbf{u}_l) = \sum_{j=0}^n \langle \mathbf{u}_l, \mathbf{n}_j \rangle D_j$ , then (16) is equivalent to requiring that

$$(17) \quad \exists \mathbf{u}_l \in M : \text{div}(\mathbf{u}_l) = \sum_{j \neq l} kb_j D_j + \langle \mathbf{u}_l, \mathbf{n}_l \rangle D_l = kD + (\langle \mathbf{u}_l, \mathbf{n}_l \rangle - kb_l) D_l.$$

The exactness of (10) ensures that (17) is equivalent to asking that

$$(18) \quad \begin{aligned} \exists \mathbf{u}_l \in M \quad : \quad d(kD + (\langle \mathbf{u}_l, \mathbf{n}_l \rangle - kb_l)D_l) &= 0 \\ \Leftrightarrow \quad k &= q'_l(kb_l - \langle \mathbf{u}_l, \mathbf{n}_l \rangle) . \end{aligned}$$

Then (18) gives that  $kD$  is Cartier if and only if  $q'_l \mid k$  for every  $0 \leq l \leq n$ . Then the inclusion  $\text{Pic}(\mathbb{P}(Q)) \hookrightarrow A_{n-1}(\mathbb{P}(Q))$  turns out to be the multiplication by  $\delta'$ . To complete the proof, notice that  $D_j$  and  $q'_j D$  give the same class in  $A_{n-1}(\mathbb{P}(Q))$  : in fact  $d(D_j - q'_j D) = 0$ . Then  $(\delta'/q'_j)D_j$  and  $\delta' D$  give the generator of  $\text{Pic}(\mathbb{P}(Q)) = \delta' A_{n-1}(\mathbb{P}(Q))$ . In particular  $(\delta'/q'_j)D_j \in |O_{\mathbb{P}(Q)}(1)|$ . This also suffices to prove that  $(\delta'/q'_j)D_j$  is ample.  $\square$

Set  $\Delta_j$  be the integral polytope associated with the divisor  $H = (\delta'/q'_j)D_j$ , as in (4). One can easily check that there exist  $n$  points  $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$ , depending on the choice of  $D_j$ , such that  $\Delta_j$  is the convex hull  $\text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$  : in particular *the ampleness of  $(\delta'/q'_j)D_j$  implies that  $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$  is a set of  $n$  linearly independent integral vectors* ([17] Corollary 2.14).

Let  $\mathfrak{P}_n$  be the set of integral polytopes in  $M_{\mathbb{R}}$  obtained as the convex hull of the origin and  $n$  linearly independent integral vectors and  $\mathfrak{F}(Q)$  be the set of fans in  $N_{\mathbb{R}}$  defining  $\mathbb{P}(Q)$ . Then we have established maps

$$(19) \quad \begin{aligned} \forall 0 \leq j \leq n, \quad \Delta_Q^j : \quad \mathfrak{F}(Q) &\longrightarrow \mathfrak{P}_n \\ \Sigma &\longmapsto \Delta_Q^j(\Sigma) := \Delta_j \end{aligned}$$

Let  $W = (w_{ik})$  be the  $n \times n$  matrix of the components of vectors  $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$  over the dual basis: namely

$$\forall k = 1, \dots, n \quad \mathbf{w}_k = \sum_{i=1}^n w_{ik} \mathbf{e}_i^{\vee}$$

where  $\{\mathbf{e}_1^{\vee}, \dots, \mathbf{e}_n^{\vee}\}$  is the dual basis of  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ . Then we get the following representation of the map  $\Delta_Q^0$ :

**THEOREM 4.** *Given the fan  $\Sigma := \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathfrak{F}(Q)$ , the image  $\Delta_Q^0(\Sigma)$  defined in (19) is the convex hull  $\text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$  of the origin with the  $n$  linearly independent integral vectors  $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$  giving the columns of the  $(0, Q)$ -weighted transverse matrix of  $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$ , i.e.*

$$W = (V^0)_Q^*,$$

where  $Q = (|V_0|, \dots, |V_n|)$ . Namely the entries of  $W$  are given by

$$\forall 1 \leq i \leq n, 1 \leq k \leq n \quad w_{ik} = \frac{\delta V_{ik}^0}{q_k V_0}$$

where  $V_{ik}^0$  is the cofactor of  $v_{ik}$  in  $V^0$  and  $V_0 = \det(V^0) = \pm q_0$  (by either (3) or (4) in Theorem 3).

*Proof.* Recalling (4), to define  $\Delta_Q^0(\Sigma) = \Delta_0$  one has to write down the hyperplanes of  $M_{\mathbb{R}}$

$$(20) \quad \forall \mathfrak{p} \in \Sigma(1) \quad \langle \mathbf{u}, \mathbf{n}_{\mathfrak{p}} \rangle = -a_{\mathfrak{p}}, \quad \text{where } \mathbf{n}_{\mathfrak{p}} \text{ generates } \mathfrak{p} \cap N,$$

for the divisor  $H = (\delta'/q'_0)D_0$ . Since  $\Sigma(1) = \{\langle \mathbf{v}_j \rangle \subset N_{\mathbb{R}} \mid j = 0, \dots, n\}$  the hyperplanes (20) are then given by

$$(21) \quad \begin{aligned} \sum_{i=1}^n n_{i0} u_i &= -\delta'/q'_0 \\ \forall k = 1, \dots, n \quad \sum_{i=1}^n n_{ik} u_i &= 0 \end{aligned}$$

where  $\mathbf{n}_j = \sum_{i=1}^n n_{ij} \mathbf{e}_i$  generates the 1-dimensional cone  $\langle \mathbf{v}_j \rangle \cap N$ . In the part (c) of Lemma 1 it has been observed that  $q'_0 \mathbf{n}_0 = -\sum_{k=1}^n q'_k \mathbf{n}_k$ . Then the first equation in (21) can be rewritten as follows

$$\sum_{i=1}^n \left( \sum_{k=1}^n q'_k n_{ik} \right) u_i = \delta'.$$

Let us represent equations in (21) by the following  $(n+1) \times (n+1)$ -matrix

$$M = \left( \begin{array}{ccc|c} \sum_{k=1}^n q'_k n_{1k} & \cdots & \sum_{k=1}^n q'_k n_{nk} & \delta' \\ n_{11} & \cdots & n_{n1} & 0 \\ & \vdots & & \vdots \\ n_{1n} & \cdots & n_{nn} & 0 \end{array} \right).$$

For  $j = 0, 1, \dots, n$ , the vertex  $\mathbf{w}_j$  of  $\Delta_Q^0(\Sigma)$  is then given by the (unique, for (3) in Theorem 3 and recalling that  $v_{ij} = d_j n_{ij}$ ) solution of the linear system associated with the matrix  $M^{j+1}$ , obtained removing the  $(j+1)$ -st row in  $M$ . Clearly  $\mathbf{w}_0 = 0$ . For  $j = k = 1, \dots, n$  we get

$$w_{ik} = M_{k+1,i} / M_{k+1,n+1}$$

where  $M_{a,b}$  is the  $(a,b)$ -cofactor in  $M$ . Observe that  $M_{k+1,n+1} = (-1)^{k-1} q'_k V_0 / a_0$  and  $M_{k+1,i} = (-1)^{k+1} \delta' d_k V_{ik}^0 / a_0$ . Then

$$w_{ik} = \frac{\delta' d_k V_{ik}^0}{q'_k V_0} = \frac{\delta' a_k d_k}{q_k} v_{ik}^* = \frac{\delta}{q_k} v_{ik}^*$$

where  $v_{ik}^* = V_{ik}^0 / V_0$  is the  $(i,k)$ -entry of  $V^{0*} := ((V^0)^{-1})^T$ . The last equality on the right is obtained by recalling Proposition 3(5) and Proposition 4.  $\square$

REMARK 7. Clearly same conclusions as in Theorem 4 can be obtained by exchanging 0 with any other value  $j$  such that  $0 \leq j \leq n$ .

REMARK 8. Let  $Q$  be a weight vector whose reduction is given by  $Q'$ . Consider  $\Sigma = \text{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathfrak{F}(Q)$  and, for any  $0 \leq j \leq n$ , consider the generator  $\mathbf{n}_j$  of the semigroup  $\langle \mathbf{v}_j \rangle \cap N$ , where  $N$  is the lattice generated by  $\mathbf{v}_0, \dots, \mathbf{v}_n$ . Then Lemma 1(c) and Theorem 3 ensure that  $\Sigma := \text{fan}(\mathbf{n}_0, \dots, \mathbf{n}_n) \in \mathfrak{F}(Q')$ . Then the previous Theorem 4 gives that

$$\Delta_Q^0(\Sigma) = \Delta_{Q'}^0(\Sigma)$$

since, recalling once again Propositions 3 and 4,

$$w_{ik} = (\delta/q_k)(V_{ik}^0/V_0) = (\delta'a/q'_k a_k)(N_{ik}^0/d_k N_0) = (\delta'/q'_k)(N_{ik}^0/N_0)$$

(here  $N$  denotes the matrix  $N = (\mathbf{n}_0, \dots, \mathbf{n}_n)$ ).

EXAMPLE 2. Let us still consider Example 1 to apply the weighted transversion and Theorem 4 for producing by hand a polytope of a given wps  $\mathbb{P}(Q)$  with the minimal polarization.

Recall that  $Q = (2, 3, 4, 15, 25)$  and the matrix fan obtained in the Example 1 is

$$V = \begin{pmatrix} -14 & 1 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 & 0 \\ -20 & 0 & 0 & 1 & 1 \\ -25 & 0 & 0 & 0 & 2 \end{pmatrix} \implies (V^0)^* = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$

Since  $\delta = \text{lcm}(2, 3, 4, 15, 25) = 300$ , we get

$$W = (V^0)^*_Q = (V^0)^* \cdot \delta I_Q = 150 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/15 & 0 \\ 0 & 0 & 0 & 1/25 \end{pmatrix}$$

$$\text{giving } W = \begin{pmatrix} 100 & 0 & 0 & 0 \\ 0 & 75 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ -50 & 0 & -10 & 6 \end{pmatrix}. \text{ Then the polytope we are looking for is}$$

$$\Delta = \text{Conv} \left( \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 0 \\ 0 \\ -50 \end{pmatrix}, \begin{pmatrix} 0 \\ 75 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 20 \\ -10 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \end{pmatrix} \right).$$

More precisely  $(\mathbb{P}_\Delta, O(1)) \cong (\mathbb{P}(Q), \delta/q_0 D_0) = (\mathbb{P}^4(2, 3, 4, 15, 25), 150 D_0)$ .

DEFINITION 6 ( $P$ -admissible matrices). A square matrix  $W \in \text{Mat}(n, n, \mathbb{Z})$  is called  $P$ -admissible if there exists an  $F$ -admissible matrix  $V \in \mathfrak{V}_n$  such that  $W$  is the weighted transverse matrix of  $V$ , which is

$$W = (V^0)^*_Q \quad \text{with} \quad Q = (|V_0|, \dots, |V_n|).$$

In other words  $W = (\mathbf{w}_1, \dots, \mathbf{w}_n)$  is admissible if and only if the polytope

$$\text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$$

belongs to the image of the map  $\Delta_Q^0$ , as defined in (19). In this case we say that  $Q, W, V$  are associated to each other.

Let us denote  $\mathfrak{M}_n \subset \text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$  the subset of  $P$ -admissible matrices: notice that any such matrix has integer entries by either Theorem 4 or the following Proposition 7.

REMARK 9. Remark 8 guarantees that weight vectors  $Q_1$  and  $Q_2$  admitting the same reduction  $Q'$  are associated with the same  $P$ -admissible matrix  $W$ , which is the  $P$ -admissible matrix associated with the reduced weight vector  $Q'$ . Conversely, there exists a unique reduced weight vector  $Q'$  to which  $W$  is associated. Proposition 9(c) will prove this fact in a purely algebraic setting; moreover Proposition 9(b) will exhibit a constructive method for finding  $Q'$ .

DEFINITION 7. Consider a matrix  $W \in \text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$ . Let  $s_i$  be the gcd of entries in the  $i$ -th row of  $\text{Adj}(W)$ . Then we define the reduced adjoint of  $W$  as follows

$$\begin{aligned} \widehat{W} &:= \frac{|\det(W)|}{\det(W)} \text{diag}\left(\frac{1}{s_1}, \dots, \frac{1}{s_n}\right) \cdot \text{Adj}(W) \\ &= \text{diag}\left(\frac{|\det(W)|}{s_1}, \dots, \frac{|\det(W)|}{s_n}\right) \cdot W^{-1} \end{aligned}$$

Notice that if  $V$  is a square matrix in  $\text{Mat}(n, n, \mathbb{Z})$  such that  $V \cdot W$  is a diagonal matrix with positive entries then

$$(22) \quad V = \text{diag}(r_1, \dots, r_n) \cdot \widehat{W}$$

for some  $r_1, \dots, r_n \in \mathbb{N}$ .

PROPOSITION 9. Let  $W$  be a  $P$ -admissible matrix and let  $Q = (q_0, \dots, q_n)$  be a reduced weight vector associated to  $W$ . Then

$$(a) \quad (\widehat{W}^T)_Q^* = W;$$

(b) if  $s := \gcd(s_1, \dots, s_n)$  is the greatest common divisor of the terms in  $\text{Adj}(W)$  then

$$q_0 = |\det(\widehat{W})|, \quad \forall 1 \leq i \leq n \quad q_i = \frac{s_i}{s}, \quad \text{lcm}(Q) = \frac{|\det(W)|}{s}$$

(c) if  $Q_1$  and  $Q_2$  are reduced weight vectors associated with the same  $P$ -admissible matrix  $W$ , then  $Q_1 = Q_2$ ;

(d) there exists a unique  $F$ -admissible matrix  $V$  associated with  $W$  and  $Q$  i.e. such that  $W = (V^0)_Q^*$  with  $Q = (|V_0|, \dots, |V_n|)$ .



*Proof.* (a).  $W$  is a  $P$ -admissible matrix. Then there exists a  $F$ -admissible matrix  $V$  such that  $W = ((V^0)^T)^{-1} \delta I_Q$  and  $Q = (|V_0|, \dots, |V_n|)$ , meaning that  $(V^0)^T W = \delta I_Q^0$  is diagonal with positive entries. Recalling (22) we get that  $(V^0)^T = \text{diag}(r_1, \dots, r_n) \cdot \widehat{W}$  for some  $r_1, \dots, r_n \in \mathbb{N}$ . But  $Q$  is reduced, which implies that the columns of  $V^0$  have coprime entries. Therefore  $r_1 = \dots = r_n = 1$  and  $(V^0)^T = \widehat{W}$ . (a) follows immediately.

(b) On the one hand  $\widehat{W} \cdot W = \text{diag}(|\det(W)|/s_1, \dots, |\det(W)|/s_n)$ . On the other hand, by (a),  $\widehat{W} = (V^0)^T$  and  $\widehat{W} \cdot W = \text{diag}(\delta/q_1, \dots, \delta/q_n)$ , where  $\delta := \text{lcm}(Q)$ . Therefore

$$(23) \quad \forall 1 \leq i \leq n \quad \frac{\delta}{q_i} = \frac{|\det(W)|}{s_i}.$$

Observe now that

$$\begin{aligned} \text{lcm}\left(\frac{\delta}{q_1}, \dots, \frac{\delta}{q_n}\right) &= \frac{\delta}{\gcd(q_1, \dots, q_n)} = \delta \\ \text{lcm}\left(\frac{|\det(W)|}{s_1}, \dots, \frac{|\det(W)|}{s_n}\right) &= \frac{|\det(W)|}{s} \end{aligned}$$

Then (23) gives that  $\delta = |\det(W)|/s$  and, for any  $1 \leq i \leq n$ ,  $q_i = s_i/s$ . Finally (a) gives that  $q_0 = |V_0| = |\det(\widehat{W})|$ .

(c) follows immediately by the previous part (b).

(d). If there exist two  $F$ -admissible matrix  $U, V$  such that they are both associated with  $W$  and  $Q$ , then

$$(\mathbf{v}_1, \dots, \mathbf{v}_n) = V^0 = U^0 = (\mathbf{u}_1, \dots, \mathbf{u}_n) \Rightarrow \mathbf{v}_0 = -\frac{1}{q_0} \sum_{i=1}^n q_i \mathbf{v}_i = -\frac{1}{q_0} \sum_{i=1}^n q_i \mathbf{u}_i = \mathbf{u}_0$$

implying that  $V = U$ .  $\square$

**REMARK 10.** In a sense the previous Proposition 9 states that, when restricted to wps fans associated with *reduced* weight vector, the weighted transversion process giving a polytope starting from a fan, can be inverted by considering the *transposed reduced adjoint* of the polytope matrix. Namely if  $W$  is a polytope matrix of  $(\mathbb{P}(Q), O(1))$ , with  $Q$  reduced, then  $V := \begin{pmatrix} \mathbf{v}_0 & | & \widehat{W}^T \end{pmatrix}$  is a fan matrix of  $\mathbb{P}(Q)$  when  $\mathbf{v}_0$  is defined by setting  $\mathbf{v}_0 = -(\sum_{i=1}^n q_i \mathbf{v}_i)/q_0$ , where  $(\mathbf{v}_1, \dots, \mathbf{v}_n) = \widehat{W}^T$ .

The following Proposition 10 shows criteria for a matrix  $W$  to be  $P$ -admissible.

**PROPOSITION 10.** Let  $W = (w_{ij}) \in \text{GL}(n, \mathbb{Q}) \cap \text{Mat}(n, n, \mathbb{Z})$  be a matrix such that  $\gcd(w_{ij}) = 1$ . Let  $s$  be the greatest common divisor of the entries in  $\text{Adj}(W)$  and  $\mathbf{v}$  be the sum of the rows of  $\text{Adj}(W)$ . Define  $q_0 = |\det(\widehat{W})|$ ,  $\delta = \frac{|\det(W)|}{s}$ . The following statements are equivalent:

- (a)  $W$  is  $P$ -admissible;
- (b) the vector  $\mathbf{v}$  is divisible by  $q_0 s$ ;

(c)  $q_0$  divides  $\delta$  and the vector  $\frac{\delta}{q_0}(1, \dots, 1)$  is in the lattice generated by the rows of  $W$ .

*Proof.* (a)  $\Rightarrow$  (b): If  $W$  is  $P$ -admissible then there exist a unique reduced weight vector  $Q$  and a unique  $F$ -admissible matrix  $V$ , associated with  $W$  like in Definition 6. By Proposition 9,  $\widehat{W} = (V^0)^T$  and  $Q = (q_0, \dots, q_n)$  with  $q_i = s_i/s$ , for  $i = 1, \dots, n$ . Let  $\mathbf{v}_i$  be the  $i$ -th row of  $\widehat{W}$ ; then  $\sum_{i=1}^n q_i \mathbf{v}_i$  is divisible by  $q_0$  since  $\widehat{W}^T = V^0$  is  $F$ -admissible, meaning that its columns satisfy the relation  $\sum_{i=1}^n q_i \mathbf{v}_i = -q_0 \mathbf{v}_0$ . Then  $\sum_{i=1}^n s_i \mathbf{v}_i$  is divisible by  $q_0 s$  and  $s_i \mathbf{v}_i$  is the  $i$ -th row of  $\text{Adj}(W)$ .

(b)  $\Rightarrow$  (a): Assume that  $q_0 s$  divides any entry in  $\mathbf{v}$ . For  $1 \leq i \leq n$ , let  $\mathbf{v}_i$  be the  $i$ -th row of  $\widehat{W}$  and  $q_i = s_i/s$  be defined as in Proposition 9(b); then  $\sum_{i=1}^n q_i \mathbf{v}_i$  is divisible by  $q_0$ . Put  $\mathbf{v}_0 = -\frac{1}{q_0} \sum_{i=1}^n q_i \mathbf{v}_i$ . Then the matrix

$$V := \left( \begin{array}{c|c} \mathbf{v}_0 & \widehat{W}^T \end{array} \right) = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n)$$

turns out to be  $F$ -admissible with respect to  $Q$  by Theorem 3(4). Then  $W = (V^0)_Q^*$  is  $P$ -admissible.

(b)  $\Leftrightarrow$  (c): the sum of the rows of  $\text{Adj}(W)$  is the row vector  $(1, \dots, 1) \cdot \text{Adj}(W) = (1, \dots, 1) \cdot \det(W) W^{-1}$ . Thus it is divisible by  $q_0 s$  if and only if there exists  $(x_1, \dots, x_n) \in \mathbb{Z}^n$  such that  $(x_1, \dots, x_n) \cdot W = \frac{\delta}{q_0}(1, \dots, 1)$ , that is if and only if (c) holds.  $\square$

### 3.2. Characterizing the polytope of a polarized wps

Given an integral polytope  $\Delta = \text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$ , for a suitable subset  $\{\mathbf{w}_1, \dots, \mathbf{w}_n\} \subset M$ , let  $W := (\mathbf{w}_1, \dots, \mathbf{w}_n)$  be the associated polytope matrix. Then the following result is a consequence of Propositions 9 and 10.

**THEOREM 5.** *Let  $\Delta = \text{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n) \subset M_{\mathbb{R}}$  be a  $n$ -dimensional integral polytope. Set  $m := \gcd(w_{ij})$  and define  $W' := \frac{1}{m}W$ . Let  $Q = (q_0, \dots, q_n)$  be the reduction of the weight vector defined as in Proposition 9(b). Then the following facts are equivalent:*

1.  $W'$  is a  $P$ -admissible matrix, hence it satisfies one of the equivalent conditions in Proposition 10,
2.  $(\mathbb{P}_{\Delta}, \mathcal{O}(1)) \cong (\mathbb{P}(Q), \mathcal{O}(m))$ .

*Proof.* (1)  $\Rightarrow$  (2): By definition if  $W'$  is  $P$ -admissible then there exists a  $F$ -admissible matrix  $V$  such that  $W' = (V^0)_{\tilde{Q}}^*$ , where  $\tilde{Q} = (|V_0|, \dots, |V_n|)$ . By Proposition 9(b)  $\tilde{Q} = Q$ . Then the polytope

$$\text{Conv}\left(\mathbf{0}, \frac{\mathbf{w}_1}{m}, \dots, \frac{\mathbf{w}_n}{m}\right)$$

belongs to the image of the map  $\Delta_Q^0$ , as defined in (19). Then  $W'$  is the polytope matrix of  $\Delta_{D'}$  for some divisor  $D' \in \mathcal{O}_{\mathbb{P}(Q)}(1)$  and  $W = mW'$  is the polytope matrix of  $\Delta := \Delta_{mD'}$  giving (2).

(2) $\Rightarrow$ (1): There exists a divisor  $D$  of  $\mathbb{P}(Q)$ , belonging to the linear system  $|O(m)|$ , such that  $\Delta = \Delta_D$ . Moreover there exists a divisor  $D' \in |O(1)|$  such that  $D = mD'$  and in particular  $\Delta = \Delta_D = m\Delta_{D'}$ . This means that  $\Delta_{D'} = \text{Conv}(\mathbf{0}, \mathbf{w}'_1, \dots, \mathbf{w}'_n)$  and  $W' := (\mathbf{w}'_1, \dots, \mathbf{w}'_n) = \frac{1}{m}W$  is a  $P$ -admissible matrix associated with  $Q$ .  $\square$

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